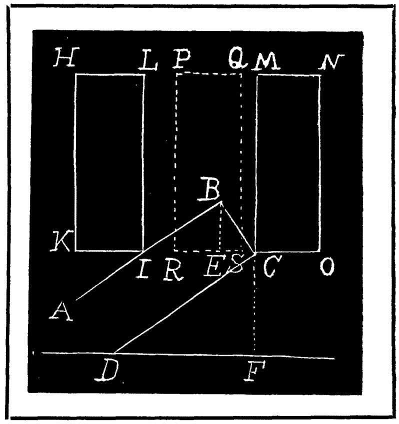
Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $HLIK$, $MNOC$ be two of the boxes, with the box $PQSR$ occupying the space between them in the position $ABCD$. Let $KH=MC=AB=a$, the length of the box; $KI=CO=BC=b$, the width of the box; $IR=SC=c$, the interval between boxes.

When the box is in the position $ABCD$ it can be taken out without removing the adjacent box or boxes.

Then CF is the width of the passageway. $CF=CD\sin CDF=a\sin\theta$.



$$IC=\frac{BC}{\cos\theta}=\frac{b}{\cos\theta}=b+2c.$$

$$\therefore \cos\theta=\frac{b}{b+2c}. \quad \therefore CF=\frac{2a}{b+2c}\sqrt{[c(b+c)]}.$$

$$\therefore \text{Width}=\frac{2a}{b+2c}\sqrt{[c(b+c)]}.$$

If the height of the box is less than the width, and IC wide enough to turn the box on its side, then write b=height and $c=\frac{1}{2}(IC-\text{height})$.

In the above $a>b+2c$, otherwise the passageway would be equal to the width of the box plus a few inches for room.

Also solved by A. H. Holmes.

167. Proposed by DR. OSWALD VEBLEN, Princeton University, Princeton, N. J.

If possible, arrange 43 objects, say the numbers 0, 1, 2, ..., in 43 sets of seven each such that every pair of objects lies in one and only one set of seven. It will then be true that two sets of seven have in common one and only one object.

Discussion by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

The following process of building successive sets of seven each leads to the conclusion that the problem is impossible. Let any set of seven be 0, 42, 41, ..., 37. As no two of this set can occur again together, each of these elements must next be combined with six sets of six elements each, these six elements being chosen from the numbers 36, 35, ..., 1. It will be shown that the twelve sets below may be taken as twelve of the desired sets additional to the above.

42, 36, 35, 34, 33, 32, 31		41, 36, 30, 24, 18, 12, 6	
42, 30, 29, 28, 27, 26, 25		41, 35, 29, 23, 17, 11, 5	
42, 24, 23, 22, 21, 20, 19	(A),	41, 34, 28, 22, 16, 10, 4	(B).
42, 18, 17, 16, 15, 14, 13		41, 33, 27, 21, 15, 9, 3	
42, 12, 11, 10, 9, 8, 7		41, 32, 26, 20, 14, 8, 2	
42, 6, 5, 4, 3, 2, 1		41, 31, 25, 19, 13, 7, 1	

In (A), 42 is any one of the first set, and the remainder of the rows must be all different both within themselves and from each other, as written.

Proceeding to (B), after choosing say, 41, from the first set, each row must contain one and only one element from each row of (A). Since the elements in the several rows of (A) are permutable, it is allowable to take the columns in (A) for the remainders of the rows of (B). For convenience call the result of omitting 42 from (A) the *square*. Further discussion is confined to this square. As the square rotated about its principal diagonal appears in (B), any *new* six must be chosen from it as one would obtain a term from a determinant. But (A) and (B) admit interchanges of both columns and rows, without losing even this reciprocal relation, so that any new six may become the principal diagonal. Hence 36, 29, 22, 15, 8, 1, may be taken as any new six, and is to be coupled with one of the elements 0, 40, 39, 38, 37, to make the fourteenth set of seven. So far 36 has been used three times, hence it must appear in four more sets of six, and the same is true of 29, 22, 15, 8, 1, while none of these six elements may appear in the same set. Thus twenty-four sets of six are accounted for in addition to the above fourteen sets of seven. The remaining sets of six, five in number, will come from the square, with the diagonal omitted.

The square possesses two transformations which are of use in reducing the labor of computing sets of six. The first is a translation along the diagonal, with provision for the edges, and is

$$\begin{matrix} (1, & 8, 15, 22, 29, 36) & (2, & 9, 16, 23, 30, 31) & (3, 10, 17, 24, 25, 32) \\ (4, & 11, 18, 19, 26, 33) & (5, & 12, 13, 20, 27, 34) & (6, \ \ 7, 14, 21, 28, 35) \end{matrix} \quad (C).$$

The second is the rotation about the diagonal. By a method to be shown, it is possible to find all the consistent sets of four sixes which contain 36. Now (C) leaves invariant the fourteen sets of seven already found, but gives by successive applications to the sets containing 36, those containing 29, then 22, etc. There are fifty-six sets of four sixes for 36, but six of them are invariant under the rotation, coupled with (42, 41), the fourteen sets of seven being also invariant. The remaining fifty sets are, by the rotation, reducible to twenty-five, so that thirty-one sets of four sixes for 36 are to be used. Of course there must be used with these, the entire fifty-six sets for 29, 22, etc.

In computing the sets for 36 the outline is this:

$$\begin{matrix} 36, 28, (19, 20, 21, 23) \\ 36, 27, (19, 20, 23) \\ 36, 26, (19, 21, 23) \\ 36, 25, (20, 21, 23) \end{matrix} \quad (D), \qquad \begin{matrix} 36, 28, 19, (14, 17) \\ 36, 27, 20, (13, 16, 17) \\ 36, 26, 21, (13, 16, 17) \\ 36, 25, 23, (14, 16) \end{matrix} \quad (E),$$

in which the parentheses in (D) indicate those elements of the third line of the square which are available for the third element of the respective rows in (D). This gives eleven partial results from (D), of which the first is

(E). In this way the total of fifty-six sets of four sixes for 36 is obtained.

The remainder of the work begins with the comparison of such of the thirty-one sets for 36 with each of the fifty-six sets for 29. Although in upwards of forty cases one obtains consistent sets of eight sixes, all of these cases fail at the trial with the sets for 22, showing that the problem is impossible.

PROBLEMS FOR SOLUTION.

ALGEBRA.

283. **Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.**

Solve $w+x+y+z=4a$, $w^2+x^2+y^2+z^2=4a^2+4b^2$, $w^3+x^3+y^3+z^3=4a^3+12ab^2$, $w^4+x^4+y^4+z^4=4a^4+4b^4+4c^4+24a^2b^2$.

GEOMETRY.

316. **Proposed by J. STEWART GIBSON, Department of Physics, Wadleigh High School, New York City.**

Determine the locus of the vertices of parabolas described by particles thrown off from the circumference of a uniformly revolving wheel.

CALCULUS.

239. **Proposed by L. H. MacDONALD, A. M., Ph. D., Sometime Tutor in the University of Cambridge, Jersey City, N. J.**

Of all triangles inscribed in a circle, find that which has the greatest perimeter.

MECHANICS.

202. **Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.**

Three equal, uniform, similar rods AB, BC, CD, freely jointed at B and C, are hung from a point by two equal strings attached at A and D. Find the position of equilibrium.

MISCELLANEOUS.

171. **Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.**

If $\lim\limits_{x=a}\dfrac{\phi(x)}{\psi(x)}=\lambda$, show $\lim\limits_{x=a}\left[\dfrac{\lambda}{\phi(x)}-\dfrac{1}{\psi(x)}\right]=\dfrac{\lambda\psi''(a)-\phi''(a)}{2\phi'(a)\psi'(a)}$.

ERRATA.

Page 97, line 10. Vol. XIII, for $x=y=w=$ etc., read $x=x_1=x_2=$ etc.
Page 98, line 1, for $G+D+U$ read $G+D+U+B$, B taken from table.
Page 97, in table add .008 to each number from 34 to 43 inclusive.